HYDRODYNAMICS OF COALESCING BINARY NEUTRON STARS: ELLIPSOIDAL TREATMENT

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ABSTRACT

We employ an approximate treatment of dissipative hydrodynamics in three dimensions to study the coalescence of binary neutron stars driven by the emission of gravitational waves. The stars are modeled as compressible ellipsoids obeying a polytropic equation of state; all internal fluid velocities are assumed to be linear functions of the coordinates. The hydrodynamic equations then reduce to a set of coupled ordinary differential equations for the evolution of the principal axes of the ellipsoids, the internal velocity parameters and the binary orbital parameters. Gravitational radiation reaction and viscous dissipation are both incorporated. We set up exact initial binary equilibrium configurations and follow the transition from the quasi-static, secular decay of the orbit at large separation to the rapid dynamical evolution of the configurations just prior to contact. A hydrodynamical instability resulting from tidal interactions significantly accelerates the coalescence at small separation, leading to appreciable radial infall velocity and tidal lag angles near contact. This behavior is reflected in the gravitational waveforms and may be observable by gravitational wave detectors under construction.

Subject headings: hydrodynamics — instabilities — stars: neutron — stars: rotation — stars: binaries: close — radiation mechanisms: gravitational

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1. INTRODUCTION

Coalescing neutron star binaries are the primary targets for the detection of gravitational waves by the planed LIGO/VIRGO laser-interferometer system (Abramovici et al. 1992). The event rate of binary coalescence has been estimated to be $\sim 10^2$ yr^{-1} Gpc^{-3} (Narayan, Piran & Shemi 1991, Phinney 1991). Extracting gravity wave signals from noise requires accurate theoretical waveforms in the frequency range $10 - 1000$ Hz, corresponding to the last few minutes of the binaries' life and orbital separations less than about 700 km (Cutler et al. 1993).

To leading order, the binary inspiral and the resulting waveform are described by Newtonian dynamics of two point masses, together with the lowest-order dissipative effect corresponding to the emission of gravitational radiation via the quadrupole formula. One important correction is the effect of post-Newtonian terms, which can produce large cumulative orbital phase error (e.g., Cutler et al. 1993), accelerate binary coalescence at small separation (Lincohn & Will 1990; Kidder, Will & Wiseman 1992, 1993), and cause precession of the orbital plane (Apostolatos et al. 1994). The other corrections come from hydrodynamical effects due to the finite size of neutron stars. The analysis of Bildsten & Cutler (1992) showed that the binary cannot be synchronized by viscous torque during the orbital decay (see also Kochanek 1992). Thus Newtonian spin-orbit coupling has negligible effect on the orbital phase at large separation, unless the neutron stars have intrinsic spins near the break-up limit (see also Lai, Rasio & Shapiro 1994a, hereafter LRS3). Another aspect of hydrodynamical binary interactions concerns resonant tidal excitations of g-modes in the stars, which occur at large orbital separation (orbital frequency $\lesssim 100$ Hz). It was shown (Reisenegger & Goldreich 1994; Lai 1994), however, that the "resonant tides" and their effects on the orbital decay rate are rather small. Therefore, it has now become clear that hydrodynamical interactions are important only during the final stage of neutron star binary coalescence, when the orbital separation is within a few stellar radii. The importance of tidal effects havs been demonstrated in our previous studies (LRS3; Lai, Rasio & Shapiro 1993b, hereafter LRS2), where it was shown that close binary systems containing sufficiently incompressible fluid, such as binary neutron stars, are dynamically unstable as a result of strong Newtonian tidal interactions. The basic consequence of the instability is the acceleration of the coalescence, leading to a significant radial infall velocity at contact, comparable to the free-fall velocity.

At large orbital separation, before the stability limit is reached, the neutron star binary evolves quasi-statically along an equilibrium sequence. This phase of the evolution has been studied in detail in LRS3, where we have considered the effects of varying neutron star masses, radii, spins and the stiffness of the equation of state. We have also shown that stable mass transfer is nearly impossible. At small orbital separation, the quasi-equilibrium treatment of LRS3 can provide qualitative features of the binary evolution. However, to obtain quantitatively accurate results when the orbital evolution takes place on a timescale comparable to the internal hydrodynamic timescale, especially when a dynamical instability is encountered, we need to use fully dynamical equations to describe the system. A new approximate energy variational formalism based on compressible ellipsoidal figures has been developed recently (Lai, Rasio & Shapiro 1994c, hereafter LRS5) to handle binary hydrodynamics. The essence of our treatment is to replace the infinite number of degrees of freedom in a fluid binary by a small number of variables specifying the essential geometric and kinematic properties of the system, such as ellipsoid axes, angular velocity and vorticity. The hydrodynamics is then described approximately by a set of ordinary differential equations (ODEs) for the time evolution of these variables, in place of the usual hydrodynamical PDEs. In this paper we apply this dynamical theory to neutron star binary evolution just prior to the final merging.

Our dynamical ellipsoid binary model represents the compressible generalization of the Riemann-Lebovitz equations for an isolated, incompressible ellipsoid (Lebovitz 1966; see also Chandrasekhar 1969, hereafter Ch69). Our approximation scheme provides an exact description of Newtonian binary stars when (a) the fluid is incompressible (polytropic index $n = 0$), and (b) the tidal effects beyond quadrupole order can be ignored. Hence for neutron stars governed by a moderately stiff equation of state ($n \lesssim 0.5$) and separated by a few stellar radii, our model is an excellent approximation to the true solution. Although our dynamical model is essentially equivalent to the affine model developed by Carter & Luminet (1983, 1986) in the context of tidal interactions with a massive black hole, our

formulation of the problem is quite different and makes explicit use of global quantities such as the total angular momentum and fluid circulation, which are conserved in the absence of dissipation. As a result, appreciable simplification in the description of many dynamical processes can be achieved. Moreover, we incorporate both viscosity and gravitational radiation reaction as possible dissipation mechanisms.

Complete understanding the final coalescence and merger of the neutron stars requires full 3D hydrodynamical simulations. So far, all simulations start from a binary configuration near contact, by which point hydrodynamical effects are already important (Oohara & Nakamura 1990; Nakamura & Oohara 1991; Shibata, Nakamura & Oohara 1992, 1993; Rasio & Shapiro 1992, 1994; Davies et al. 1994). Modeling the distant, pre-contact phase of binary coalescence by 3D numerical codes would require prohibitively large amounts of computer resources. Our formulation of binary dynamics in terms of ODEs allows us to follow the evolution of the system over a large number of orbits without having to worry about excessive computational time or about the growth of numerical errors. Thus our dynamical binary model is particularly useful in studying the pre-contact transition phase in which the binary evolves from the quasi-static secular regime to the fully dynamical regime. In addition, our model can provide reliable initial data for 3D simulations of the binary merger and can serve as a check of more sophisticated numerical routines.

The main purpose of this paper is to present the complete dynamical equations for two stars modeled as ellipsoids in binary orbit (§2), including dissipative forces of viscosity and gravitational radiation reaction (§3). The equations are used to demonstrate the dynamical instability in the binary (§4) and to study a few selected scenarios for neutron star binary coalescence driven by gravitational radiation (§5). No attempt is made at a complete survey of parameter space. Instead, we show how straightforward it is to vary the individual masses, equation of state and spins of the interacting stars so that our method can be easily employed by future investigators to examine other cases of interest.

We adopt geometrized units and set $G = c = 1$ throughout the paper.

2. DYNAMICAL EQUATIONS OF DARWIN-RIEMANN BINARIES

The dynamical equations for compressible Roche-Riemann binaries, consisting of a finite-size star and a point mass, have been derived in LRS5. The readers are referred to that paper for the basic definitions of variables, notation and further details. The generalization to compressible *Darwin-Riemann binaries*, consisting of two finite-size stars, is straightforward and is sketched below.

Consider a binary system containing two stars of mass M and M', each obeying a polytropic equation of state. Throughout this paper unprimed quantities refer to the star of mass M and primed quantities refer to the star of mass M'. The density and pressure are related by the relation

$$P = K\rho^{1+1/n}, \quad P' = K'\rho'^{1+1/n'}. \tag{2.1}$$

Note that for given n and n', the values of K and K' are determined from the equilibrium radii R_o and R_o' of the nonrotating stars with the same masses. Thus any realistic equation of state can be approximately fit by a polytropic law (cf. LRS3, §4.1).

In the Darwin-Riemann binary model, both stars have the structure of compressible *Riemann-S ellipsoids* (Ch69, Lai, Rasio & Shapiro 1993a, hereafter LRS1). A general Riemann-S ellipsoid is characterized by the angular velocity $\mathbf{\Omega} = \Omega\mathbf{e}_z$ *of the ellipsoidal figure* (the pattern speed) about a principal axis and the internal motion of the fluid with uniform vorticity $\zeta = \zeta\mathbf{e}_z$ along the same axis (in the frame corotating with the figure). We assume that the surfaces of constant density inside the star form self-similar ellipsoids. The number of degrees of freedom for star M is then reduced from infinity to five: three principal axes a_1, a_2, a_3, and two angles ϕ, ψ, defined such that $d\phi/dt = \Omega$ and $d\psi/dt = \Lambda$, where $\Lambda = -a_1 a_2 \zeta/(a_1^2 + a_2^2)$. Similarly, for star M', we have a_1', a_2', a_3', ϕ' and ψ'. The spins are assumed to be aligned parallel to the orbital angular momentum. The orbit is specified by the orbital separation r and the true anomaly θ (see Fig. 1). Thus, a total of 12 variables completely specify the dynamics of the binary system.

The Lagrangian of the system can be written as the sum of the intrinsic stellar components L_s and L_s', and the orbital component L_{orb} according to

$$L = L_s + L_s' + L_{orb}. \tag{2.2}$$

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To derive the stellar term L_s, let $\mathbf{e}_1$, $\mathbf{e}_1$ and $\mathbf{e}_3$ be the basis unit vectors along the instantaneous direction of the principal axes of the ellipsoid M, with $\mathbf{e}_3$ perpendicular to the orbital plane (the "*body frame*"). In the inertial frame, the velocity field $\mathbf{u}$ of the fluid inside M relative to the center of mass can be written as

$$\mathbf{u} = \left[\left(\frac{a_1}{a_2}\Lambda - \Omega \right) x_2 \mathbf{e}_1 + \left(-\frac{a_2}{a_1}\Lambda + \Omega \right) x_1 \mathbf{e}_2 \right] + \left[\frac{\dot{a}_1}{a_1} x_1 \mathbf{e}_1 + \frac{\dot{a}_2}{a_2} x_2 \mathbf{e}_2 + \frac{\dot{a}_3}{a_3} x_3 \mathbf{e}_3 \right]. \tag{2.3}$$

The kinetic energy of M relative to its center of mass is then given by

$$T = \frac{1}{2} I(\Lambda^2 + \Omega^2) - \frac{2}{5}\kappa_n M a_1 a_2 \Lambda\Omega + \frac{1}{10}\kappa_n M(\dot{a}_1^2 + \dot{a}_2^2 + \dot{a}_3^3), \tag{2.4}$$

where $I = \kappa_n M(a_1^2 + a_2^2)/5$ and $\kappa_n \leq 1$ is a constant of order unity which depends only on n (see Table I in LRS1).

The internal energy of M is given by

$$U = \int n \frac{P}{\rho} dm = k_1 K \rho_c^{1/n} M. \tag{2.5}$$

where k_1 is another constant depending only on n and $\rho_c \propto 1/(a_1 a_2 a_3)$ is the central density. The self-gravitational potential energy is given by

$$W = -\frac{3}{5-n} \frac{M^2}{R} \frac{\mathcal{I}}{2R^2}, \quad \text{with} \quad \mathcal{I} = A_1 a_1^2 + A_2 a_2^2 + A_3 a_3^2, \tag{2.6}$$

where $R \equiv (a_1 a_2 a_3)^{1/3}$ is the mean radius of the ellipsoid, and dimensionless index symbols A_i are defined as in Ch69 (§17). Therefore we have

$$L_s = T - U - W. \tag{2.7}$$

Similar expressions for M' can also be derived. The orbital component is clearly

$$L_{orb} = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\theta}^2 - W_i, \tag{2.8}$$

where $\mu = MM'/(M+M')$ is the reduced mass. The gravitational interaction energy W_i between M and M' is given to quadrupole order by

$$\begin{aligned} W_i = &-\frac{MM'}{r} - \frac{M'}{2r^3}[I_{11}(3\cos^2\alpha - 1) + I_{22}(3\sin^2\alpha - 1) - I_{33}] \\ &-\frac{M}{2r^3}[I_{11}'(3\cos^2\alpha' - 1) + I_{22}'(3\sin^2\alpha' - 1) - I_{33}'], \end{aligned} \tag{2.9}$$

where $\alpha = \theta - \phi$, $\alpha' = \theta - \phi'$, and $I_{ij} = \kappa_n M a_i^2 \delta_{ij}/5$, and similarly for I_{ij}'.

Given the Lagrangian, the dynamical equations can then be obtained from the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}, \tag{2.10}$$

where $\{q_i\} = \{a_i, \phi, \psi, a_i', \phi', \psi', r, \theta\}$. For $q_i = \phi$, we have

$$\frac{dJ_s}{dt} = \frac{3M'}{2r^3}\sin 2\alpha(I_{11} - I_{22}) = \mathcal{N}, \tag{2.11}$$

where J_s is the "spin" angular momentum of M

$$J_s = \frac{\partial L}{\partial \Omega} = I\Omega - \frac{2}{5}\kappa_n M a_1 a_2 \Lambda, \tag{2.12}$$

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and $\mathcal{N}$ is the tidal torque on the star. Similarly, for $q_i = \phi'$, we have

$$\frac{dJ'_s}{dt} = \frac{3M}{2r^3} \sin 2\alpha' (I'_{11} - I'_{22}) = \mathcal{N}'. \tag{2.13}$$

For $q_i = \theta$, equation (2.13) gives

$$\frac{dJ_{orb}}{dt} = -\mathcal{N} - \mathcal{N}', \tag{2.14}$$

where $J_{orb} = \mu r^2 \dot{\theta}$ is the orbital angular momentum. Thus the total angular momentum

$$J = J_s + J'_s + J_{orb}, \tag{2.15}$$

is conserved. Finally, for $q_i = \psi$ and $q_i = \psi'$, we obtain

$$\frac{d\mathcal{C}}{dt} = 0, \qquad \frac{d\mathcal{C}'}{dt} = 0, \tag{2.16}$$

where $\mathcal{C}$ is the fluid circulation in star M

$$\mathcal{C} = \frac{\partial L}{\partial \Lambda} = I\Lambda - \frac{2}{5}\kappa_n M a_1 a_2 \Omega, \tag{2.17}$$

and similarly for $\mathcal{C}'$. Thus the fluid circulations in M and M' are individually conserved.

The complete dynamical equations can be written in a numerically convenient form as follows:

$$\ddot{a}_1 = a_1(\Omega^2 + \Lambda^2) - 2a_2\Omega\Lambda - \frac{2\pi}{q_n}a_1 A_1 \bar{\rho} + \left(\frac{5k_1}{n\kappa_n}\frac{P_c}{\rho_c}\right)\frac{1}{a_1} + \frac{M'a_1}{r^3}(3\cos^2\alpha - 1), \tag{2.18}$$

$$\ddot{a}_2 = a_2(\Omega^2 + \Lambda^2) - 2a_1\Omega\Lambda - \frac{2\pi}{q_n}a_2 A_2 \bar{\rho} + \left(\frac{5k_1}{n\kappa_n}\frac{P_c}{\rho_c}\right)\frac{1}{a_2} + \frac{M'a_2}{r^3}(3\sin^2\alpha - 1), \tag{2.19}$$

$$\ddot{a}_3 = -\frac{2\pi}{q_n}a_3 A_3 \bar{\rho} + \left(\frac{5k_1}{n\kappa_n}\frac{P_c}{\rho_c}\right)\frac{1}{a_3} - \frac{M'a_3}{r^3}, \tag{2.20}$$

$$\dot{\Omega} = \left(\frac{a_2}{a_1} - \frac{a_1}{a_2}\right)^{-1}\left[2\left(\frac{\Omega}{a_2} + \frac{\Lambda}{a_1}\right)\dot{a}_1 - 2\left(\frac{\Omega}{a_1} + \frac{\Lambda}{a_2}\right)\dot{a}_2 - \frac{3M'}{2r^3}\left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right)\sin 2\alpha\right], \tag{2.21}$$

$$\dot{\Lambda} = \left(\frac{a_2}{a_1} - \frac{a_1}{a_2}\right)^{-1}\left[2\left(\frac{\Omega}{a_1} + \frac{\Lambda}{a_2}\right)\dot{a}_1 - 2\left(\frac{\Omega}{a_2} + \frac{\Lambda}{a_1}\right)\dot{a}_2 - \frac{3M'}{r^3}\sin 2\alpha\right], \tag{2.22}$$

$$\ddot{r} = r\dot{\theta}^2 - \frac{M + M'}{r^2} - \frac{3\kappa_n}{10}\frac{M + M'}{r^4}\left[a_1^2(3\cos^2\alpha - 1) + a_2^2(3\sin^2\alpha - 1) - a_3^2\right]$$
$$- \frac{3\kappa'_n}{10}\frac{M + M'}{r^4}\left[a_1'^2(3\cos^2\alpha' - 1) + a_2'^2(3\sin^2\alpha' - 1) - a_3'^2\right], \tag{2.23}$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} - \frac{3\kappa_n}{10}\frac{M + M'}{r^5}(a_1^2 - a_2^2)\sin 2\alpha - \frac{3\kappa'_n}{10}\frac{M + M'}{r^5}(a_1'^2 - a_2'^2)\sin 2\alpha', \tag{2.24}$$

where $\dot{\theta} = \Omega_{orb}$, $\dot{\phi} = \Omega$, and $q_n \equiv \kappa_n(1 - n/5)$. The equations for $\ddot{a}_i$, $\dot{\Omega}'$ and $\dot{\Lambda}'$ can be similarly written down by switching unprimed quantities with primed quantities in equations (2.18)-(2.24). Also, the pressure term $(5k_1 P_c)/(n\kappa_n \rho_c)$ can be conveniently expressed in terms of R_o, M and other dynamical variables as

$$\frac{5k_1}{n\kappa_n}\frac{P_c}{\rho_c} = \frac{M}{q_n R_o}\left(\frac{R_o}{R}\right)^{3/n}, \qquad (n \neq 0), \tag{2.25}$$

(see LRS5, where the equivalent expressions for the limiting case of $n = 0$ can also be found).

3. GRAVITATIONAL RADIATION REACTION AND VISCOUS DISSIPATION

Dissipation modifies Euler-Lagrange equations according to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} + \mathcal{F}_{q_i},\tag{3.1}$$

where $\mathcal{F}_{q_i}$ is the "generalized" force associated with variable q_i, and it is defined via the non-conservative dissipation rate $\mathcal{W}$ (Rayleigh's dissipation function, cf. Goldstein 1980) by $\mathcal{W} = \mathcal{F}_{q_i}\dot{q}_i$.

3.1 Gravitational Radiation Reaction

The main driving force for neutron star binary coalescence is gravitational radiation reaction. Consider the *orbital coordinates* $\{\mathbf{e}_{\bar{i}}\}$, centered at the center-of-mass (CM) of the system with $\mathbf{e}_{\bar{1}}$ along the line joining MM', $\mathbf{e}_{\bar{3}}$ perpendicular to the orbital plane, and $\mathbf{e}_{\bar{2}}$ perpendicular to $\mathbf{e}_{\bar{1}}$ and $\mathbf{e}_{\bar{3}}$ (see Fig. 1). This coordinate basis rotates with angular velocity Ω_{orb} with respect to an inertial coordinate system. In the weak-field, slow motion regime of general relativity, the gravitational radiation emission induces a back-reaction potential Φ_{react}, which can be written as (Misner, Thorne & Wheeler 1973):

$$\Phi_{react} = \frac{1}{5}\mathfrak{I}^{(5)}_{\bar{i}\bar{j}} x_{\bar{i}} x_{\bar{j}}.\tag{3.2}$$

where $\mathfrak{I}^{(5)}_{\bar{i}\bar{j}}$ is the fifth derivative of the reduced quadrupole moment tensor of the system *projected onto the orbital frame*. The contribution from M to $\mathcal{W}$ is given by

$$\begin{aligned}
\mathcal{W}_M ={}& -\int_M \mathbf{v}\cdot\nabla\Phi_{react}\,dm\\
={}& -\frac{2}{5}\left(\frac{1}{5}\kappa_n M\right)\left[\left(\mathfrak{I}^{(5)}_{\bar{1}\bar{1}}\cos^2\alpha + \mathfrak{I}^{(5)}_{\bar{2}\bar{2}}\sin^2\alpha - \mathfrak{I}^{(5)}_{\bar{1}\bar{2}}\sin 2\alpha\right)a_1\dot{a}_1\right.\\
&+\left(\mathfrak{I}^{(5)}_{\bar{1}\bar{1}}\sin^2\alpha + \mathfrak{I}^{(5)}_{\bar{2}\bar{2}}\cos^2\alpha + \mathfrak{I}^{(5)}_{\bar{1}\bar{2}}\sin 2\alpha\right)a_2\dot{a}_2 + \mathfrak{I}^{(5)}_{\bar{3}\bar{3}}a_3\dot{a}_3\\
&\left.+\left(\mathfrak{I}^{(5)}_{\bar{1}\bar{2}}\cos 2\alpha + (\mathfrak{I}^{(5)}_{\bar{1}\bar{1}} - \mathfrak{I}^{(5)}_{\bar{2}\bar{2}})\frac{1}{2}\sin 2\alpha\right)(a_1^2 - a_2^2)\Omega\right]\\
&-\frac{2}{5}M\left(\mathfrak{I}^{(5)}_{\bar{1}\bar{1}}r_{cm}\dot{r}_{cm} + \mathfrak{I}^{(5)}_{\bar{1}\bar{2}}r_{cm}^2\Omega_{orb}\right),
\end{aligned}\tag{3.3}$$

where $\mathbf{v} = \mathbf{u} + \mathbf{u}_{orb}$ is the fluid velocity in M, r_{cm} is the distance from the CM of the system to the CM of M. A similar expression can be written down for M'. Therefore

$$\begin{aligned}
\mathcal{W} ={}& \mathcal{W}_M + \mathcal{W}_{M'}\\
={}& -\frac{2}{5}\left(\frac{1}{5}\kappa_n M\right)\left[\left(\mathfrak{I}^{(5)}_{\bar{1}\bar{1}}\cos^2\alpha + \mathfrak{I}^{(5)}_{\bar{2}\bar{2}}\sin^2\alpha - \mathfrak{I}^{(5)}_{\bar{1}\bar{2}}\sin 2\alpha\right)a_1\dot{a}_1\right.\\
&+\left(\mathfrak{I}^{(5)}_{\bar{1}\bar{1}}\sin^2\alpha + \mathfrak{I}^{(5)}_{\bar{2}\bar{2}}\cos^2\alpha + \mathfrak{I}^{(5)}_{\bar{1}\bar{2}}\sin 2\alpha\right)a_2\dot{a}_2 + \mathfrak{I}^{(5)}_{\bar{3}\bar{3}}a_3\dot{a}_3\\
&\left.+\left(\mathfrak{I}^{(5)}_{\bar{1}\bar{2}}\cos 2\alpha + (\mathfrak{I}^{(5)}_{\bar{1}\bar{1}} - \mathfrak{I}^{(5)}_{\bar{2}\bar{2}})\frac{1}{2}\sin 2\alpha\right)(a_1^2 - a_2^2)\Omega\right]\\
&-\frac{2}{5}\left(\frac{1}{5}\kappa'_n M'\right)\left[\left(\mathfrak{I}^{(5)}_{\bar{1}\bar{1}}\cos^2\alpha' + \mathfrak{I}^{(5)}_{\bar{2}\bar{2}}\sin^2\alpha' - \mathfrak{I}^{(5)}_{\bar{1}\bar{2}}\sin 2\alpha'\right)a'_1\dot{a}'_1\right.\\
&+\left(\mathfrak{I}^{(5)}_{\bar{1}\bar{1}}\sin^2\alpha' + \mathfrak{I}^{(5)}_{\bar{2}\bar{2}}\cos^2\alpha' + \mathfrak{I}^{(5)}_{\bar{1}\bar{2}}\sin 2\alpha'\right)a'_2\dot{a}'_2 + \mathfrak{I}^{(5)}_{\bar{3}\bar{3}}a'_3\dot{a}'_3\\
&\left.+\left(\mathfrak{I}^{(5)}_{\bar{1}\bar{2}}\cos 2\alpha' + (\mathfrak{I}^{(5)}_{\bar{1}\bar{1}} - \mathfrak{I}^{(5)}_{\bar{2}\bar{2}})\frac{1}{2}\sin 2\alpha'\right)(a'^2_1 - a'^2_2)\Omega'\right]\\
&-\frac{2}{5}\mu\left(\mathfrak{I}^{(5)}_{\bar{1}\bar{1}}r\dot{r} + \mathfrak{I}^{(5)}_{\bar{1}\bar{2}}r^2\Omega_{orb}\right).
\end{aligned}\tag{3.4}$$

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The dissipative forces due to gravitational radiation are then given by $\mathcal{F}_{q_i} = \partial \mathcal{W}/\partial \dot{q}_i$. Clearly, $\mathcal{F}_\psi = \mathcal{F}_{\psi'} = 0$, thus gravitational radiation reaction conserves $\mathcal{C}$ and $\mathcal{C}'$. With the inclusion of gravitational radiation reaction forces, the dynamical equations (2.18)-(2.24) for binaries are modified according to:

$$\ddot{a}_1 = \{\cdots\} - \frac{2}{5} \left[\mathbb{I}_{\bar{1}\bar{1}}^{(5)} \cos^2\alpha + \mathbb{I}_{\bar{2}\bar{2}}^{(5)} \sin^2\alpha - \mathbb{I}_{\bar{1}\bar{2}}^{(5)} \sin 2\alpha \right] a_1, \tag{3.5}$$

$$\ddot{a}_2 = \{\cdots\} - \frac{2}{5} \left[\mathbb{I}_{\bar{1}\bar{1}}^{(5)} \sin^2\alpha + \mathbb{I}_{\bar{2}\bar{2}}^{(5)} \cos^2\alpha + \mathbb{I}_{\bar{1}\bar{2}}^{(5)} \sin 2\alpha \right] a_2, \tag{3.6}$$

$$\ddot{a}_3 = \{\cdots\} - \frac{2}{5} \mathbb{I}_{\bar{3}\bar{3}}^{(5)} a_3, \tag{3.7}$$

$$\dot{\Omega} = \left(\frac{a_2}{a_1} - \frac{a_1}{a_2} \right)^{-1} \left[\{\cdots\} + \frac{2}{5} \left(\mathbb{I}_{\bar{1}\bar{2}}^{(5)} \cos 2\alpha + \frac{1}{2}(\mathbb{I}_{\bar{1}\bar{1}}^{(5)} - \mathbb{I}_{\bar{2}\bar{2}}^{(5)}) \sin 2\alpha \right) \left(\frac{a_1}{a_2} + \frac{a_2}{a_1} \right) \right], \tag{3.8}$$

$$\dot{\Lambda} = \left(\frac{a_2}{a_1} - \frac{a_1}{a_2} \right)^{-1} \left[\{\cdots\} + \frac{4}{5} \left(\mathbb{I}_{\bar{1}\bar{2}}^{(5)} \cos 2\alpha + \frac{1}{2}(\mathbb{I}_{\bar{1}\bar{1}}^{(5)} - \mathbb{I}_{\bar{2}\bar{2}}^{(5)}) \sin 2\alpha \right) \right], \tag{3.9}$$

$$\ddot{r} = \{\cdots\} - \frac{2}{5} \mathbb{I}_{\bar{1}\bar{1}}^{(5)} r, \tag{3.10}$$

$$\ddot{\theta} = \{\cdots\} - \frac{2}{5} \mathbb{I}_{\bar{1}\bar{2}}^{(5)}, \tag{3.11}$$

where $\{\cdots\}$ denote the nondissipative terms that already exist in equations (2.18)–(2.24) (This notation will be used throughout the paper). The expressions for $\ddot{a}_i'$, $\dot{\Omega}'$ and $\dot{\Lambda}'$ are similar.

When the timescale for both orbital and internal quantities to change is much longer than the rotation period, e.g., $|da_i/dt| << |\Omega a_i|$, simple expressions for $\mathbb{I}_{ij}^{(5)}$ can be derived. To order da_i/dt, the nontrivial components of $\mathbb{I}_{\bar{i}\bar{j}}^{(5)}$ are

$$
\begin{aligned}
\mathbb{I}_{\bar{1}\bar{1}}^{(5)} = -\mathbb{I}_{\bar{2}\bar{2}}^{(5)} &= 16\Omega^5(I_{11} - I_{22})\sin 2\alpha + 16\Omega'^5(I_{11}' - I_{22}')\sin 2\alpha' \\
&+ 40\Omega_{orb}^3[2\Omega_{orb}\mu r\dot{r} + 2\dot{\Omega}\mu r^2] + 40\Omega^3[\Omega(\dot{I}_{11} - \dot{I}_{22}) + 2\dot{\Omega}(I_{11} - I_{22})]\cos 2\alpha \\
&+ 40\Omega'^3[\Omega'(\dot{I}_{11}' - \dot{I}_{22}') + 2\dot{\Omega}'(I_{11}' - I_{22}')]\cos 2\alpha', \\
\mathbb{I}_{\bar{1}\bar{2}}^{(5)} = \mathbb{I}_{\bar{2}\bar{1}}^{(5)} &= 16\Omega_{orb}^5\mu r^2 + 16\Omega^5(I_{11} - I_{22})\cos 2\alpha + 16\Omega'^5(I_{11}' - I_{22}')\cos 2\alpha' \\
&- 40\Omega^3[\Omega(\dot{I}_{11} - \dot{I}_{22}) + 2\dot{\Omega}(I_{11} - I_{22})]\sin 2\alpha \\
&- 40\Omega'^3[\Omega'(\dot{I}_{11}' - \dot{I}_{22}') + 2\dot{\Omega}'(I_{11}' - I_{22}')]\sin 2\alpha'.
\end{aligned}
\tag{3.12}
$$

These generalize the expressions derived in Appendix A of LRS5 for Roche-Riemann binaries.

3.2 Viscous Dissipation

Viscous dissipation forces can be easily incorporated into our dynamical equations. Since the motion of the center of mass of the star is not affected by viscous dissipation (which depends only on the shear stresses inside the star), the viscous forces for an isolated star as derived in LRS5 (§4.1) can be directly applied to binaries. The dissipation rate due to shear viscosity is $\mathcal{W} = \mathcal{W}_M + \mathcal{W}_{M'}$, with

$$
\begin{aligned}
\mathcal{W}_M = -\frac{4}{3}\bar{\nu}M &\left[\left(\frac{\dot{a}_1}{a_1} \right)^2 + \left(\frac{\dot{a}_2}{a_2} \right)^2 + \left(\frac{\dot{a}_3}{a_3} \right)^2 - \left(\frac{\dot{a}_1}{a_1} \right)\left(\frac{\dot{a}_2}{a_2} \right) - \left(\frac{\dot{a}_1}{a_1} \right)\left(\frac{\dot{a}_3}{a_3} \right) \right. \\
&\left. - \left(\frac{\dot{a}_2}{a_2} \right)\left(\frac{\dot{a}_3}{a_3} \right) \right] - \bar{\nu}M\Lambda^2 \left(\frac{a_1^2 - a_2^2}{a_1 a_2} \right)^2,
\end{aligned}
\tag{3.13}
$$

where $\bar{\nu}$ is the mass-averaged shear kinematic viscosity $\bar{\nu} = \int \nu\, dm/M$. The expression for $\mathcal{W}_{M'}$ can be written down similarly. Since $\mathcal{W}$ is quadratic in $\dot{q}_i$, the dissipative forces are given by $\mathcal{F}_{q_i} =$

$(1/2)\partial\mathcal{W}/\partial\dot{q}_i$. Clearly, in the presence of viscosity, the fluid circulations $\mathcal{C}$ and $\mathcal{C}'$ are not conserved. However, viscous forces do not affect angular momentum, i.e., $\mathcal{F}_\theta = \mathcal{F}_\phi = \mathcal{F}_{\phi'} = 0$. The dynamical equations (2.18)-(2.24) are modified according to

$$\ddot{a}_1 = \{\cdots\} - \frac{10}{3\kappa_n}\bar{\nu}\left(\frac{2\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_3}{a_3}\right)\frac{1}{a_1}, \tag{3.14}$$

$$\ddot{a}_2 = \{\cdots\} - \frac{10}{3\kappa_n}\bar{\nu}\left(\frac{2\dot{a}_2}{a_2} - \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_3}{a_3}\right)\frac{1}{a_2}, \tag{3.15}$$

$$\ddot{a}_3 = \{\cdots\} - \frac{10}{3\kappa_n}\bar{\nu}\left(\frac{2\dot{a}_3}{a_3} - \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2}\right)\frac{1}{a_3}, \tag{3.16}$$

$$\dot{\Omega} = \left(\frac{a_2}{a_1} - \frac{a_1}{a_2}\right)^{-1}\left[\{\cdots\} + \frac{10}{\kappa_n}\bar{\nu}\frac{a_1^2 - a_2^2}{a_1^2 a_2^2}\Lambda\right], \tag{3.17}$$

$$\dot{\Lambda} = \left(\frac{a_2}{a_1} - \frac{a_1}{a_2}\right)^{-1}\left[\{\cdots\} + \frac{5}{\kappa_n}\bar{\nu}\frac{a_1^2 - a_2^2}{a_1 a_2}\left(\frac{1}{a_1^2} + \frac{1}{a_2^2}\right)\Lambda\right], \tag{3.18}$$

and the expressions for $\ddot{a}_i'$, $\dot{\Omega}'$ and $\dot{\Lambda}'$ are similar, while the expressions for $\ddot{r}$ and $\ddot{\theta}$ are unchanged.

4. DYNAMICAL INSTABILITY IN DARWIN-RIEMANN BINARIES

The dynamical instability in a binary system results from tidal interactions between the two stars. The height h of the tidal bulge raised on star M by its companion M' is of order $h \sim R(M'/M)(R/r)^3$. This tidal deformation makes the gravitational interaction between M and M' more attractive, and the tidal potential energy is

$$W_{tide} \sim -\frac{M'Q}{r^3} \sim -\kappa_n\frac{M'^2 R^5}{r^6}, \tag{4.1}$$

(see eq. [2.9]), where $Q \sim \kappa_n MRh$ is the quadrupole moment of M. Thus at sufficiently small binary separation, assuming κ_n is not too small (i.e., the star is not too compressible), the binary interaction potential energy $\sim (-MM'/r + W_{tide})$ becomes steeper than the point-mass contribution $-MM'/r$. This is the cause of the dynamical instability in the binary, as is common to all interaction potentials that are sufficiently steeper than $1/r$ (cf. Goldstein 1980, §3.6).

To determine the dynamical stability limit, one only needs to construct a sequence of equilibrium models with constant $\mathcal{C}$ and $\mathcal{C}'$. The onset of dynamical instability corresponds to the turning point in the energy curve along the sequence. This procedure is described and applied in LRS1 and LRS4, where numerical values of the stability limits for various binary models have been tabulated.

Using our dynamical equations with no radiation reaction or viscosity, we can now study how the instability develops in time. In Figure 2 we show an example of the time evolution of an unstable system with $n = n' = 0.5$, $K = K'$, $p = M/M' = 1/2$ and $\Lambda = \Lambda' = 0$ (corotation). The dynamical stability limit is at $\hat{r} = r/(a_1 + a_1') = 1.174$ (cf. LRS4, Table 2). The dynamical equations are integrated numerically using a standard fifth-order Runge-Kutta scheme with adaptive stepsize (Press et al 1992). At $t = 0$, an unstable equilibrium solution is constructed for $\hat{r} = r/(a_1 + a_1') = 1.17$. This equilibrium solution is then perturbed by setting $\dot{r} = -10^{-4}(M/R_o)^{1/2}$. For comparison, the results of an integration for a stable binary with $\hat{r} = 1.18$, and with the same applied perturbation are also shown. We see clearly that as the dynamical instability develops, a_1 increases while r decreases, and this is accompanied by the significant development of tidal lag angle in the two stars (α, $\alpha' > 0$) and de-synchronization (Λ, $\Lambda > 0$). Of course, the precise evolution of an unstable binary depends on how the initial configuration is perturbed.

The development of tidal lag in the absence of fluid dissipation can be qualitatively understood as follows (cf. Lai 1994). For star M in the binary system, the tidal potential $\propto 1/r^3$ due to the companion M' acts like an external perturbing force, with a driving frequency $\sim \Delta\Omega = \Omega_{orb} - \Omega_s$, where Ω_{orb} and Ω_s are the orbital and spin angular frequencies. The star has an intrinsic dynamical frequency of order $\omega_o \sim (M/R_o^2)^{1/2}$. Schematically, the equation governing the tidal distortion ξ can be written as

$$\ddot{\xi} + \omega_o^2\xi \propto \frac{1}{r^3}e^{i\Delta\Omega t}. \tag{4.2}$$

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If r and $\Delta\Omega$ were constant in time, then the stationary tide would be in phase with the driving force. However, when r and/or Ω_{orb} change(s) during binary evolution, we have (assuming $\omega_o >> \Delta\Omega$)

$$\xi \propto \frac{1}{\omega_o^2 r^3 (1 - i\alpha_{dyn})} e^{i\Delta\Omega t}, \tag{4.3}$$

where the lag angle is given by

$$\alpha_{dyn} \sim \frac{\Delta\Omega}{\omega_o^2 t_d}, \tag{4.4}$$

and $t_d \sim |r/\dot{r}| \sim |\Omega_{orb}/\dot{\Omega}_{orb}|$ is the orbital decay timescale. Thus a *dynamical tidal lag* arises naturally even without fluid dissipation, and it is due to the finite time necessary for the star to adjust its structure to the rapidly changing tidal potential. Since the orbit decays rapidly as a result of dynamical instability, the binary de-synchronizes, and thus α_{dyn} becomes large at small r.

5. NEUTRON STAR BINARY COALESCENCE

The main parameters that enter the evolution equations of §2 and 3 are the mass ratio $p = M/M'$, and the ratios R_o/M and R_o'/M'. The ratio R_o/M is determined from the nuclear equation of state (EOS). For the canonical neutron star mass $M = 1.4 M_\odot$, all EOS's tabulated in Arnett & Bowers (1977) give R_o/M in the range of 4–8. Recent microscopic nuclear calculations indicate that the neutron star radius R_o is $\simeq 10$ km, almost independent of the mass for M in the range of $0.8 M_\odot$ to $1.5 M_\odot$ (Wiringa, Fiks & Fabrocini 1988). Thus we will use $R_o/M \simeq 5$ as a representative value and assume $R_o = R_o'$.

A polytrope is only an approximate parametrization for a real EOS. We can determine the approximate polytropic index n that mimics the structure of a real neutron star by using the tabulated values of the moment of inertia of neutron stars (see LRS3, §4.1). Typically we find $n \simeq 0.5$ for $M \simeq 1.4 M_\odot$ for the EOS of Wiringa, Fiks & Fabrocini (1988).

Other parameters needed for the calculations are the spins of the two neutron stars Ω_s and Ω_s' when the binary separation is large. For a uniformly rotating neutron star, the spin rate is limited by mass-shedding, $\bar{\Omega}_s \equiv \Omega_s/(M/R_o^3)^{1/2} \lesssim 0.6$ (Friedman, Ipser & Parker 1986, Cook, Shapiro & Teukolsky 1992). At large orbital separation, we have $a_1 \to a_2$ and $\Omega \to \Omega_{orb} \to (M+M')^{1/2}/r^{3/2} <<$ Ω_s, thus $J_s \to -I\Lambda$ and $\mathcal{C} \to I\Lambda$ (see eqs. [2.12], [2.17]). The fluid circulation inside the star, which is conserved during the binary evolution in the absence of viscosity, is therefore given by $\mathcal{C} = -I\Omega_s$. Here we identify $\Omega_s = -\Lambda(r = \infty)$ as the spin angular velocity at large r (for an axisymmetric spheroid, uniform spin and vorticity are indistinguishable; cf. eq. [2.3]). Note that when Ω_s is positive (the spin is in the same direction as the orbital angular momentum), $\mathcal{C}$ is negative. For simplicity we assume $\Omega_s' = 0 = \mathcal{C}'$.

In Figure 3 we show two examples of the pre-contact evolution of binary neutron stars. Both calculations start at $t = 0$ with $r = 5R_o$, and terminate when the surfaces of the stars contact. The coalescence is driven by gravitational radiation reaction, and we ignore viscosity for now. In the first example, the stars have equal masses, with $R_o/M = R_o'/M' = 5$, $n = n' = 0.5$, and both have zero spin $\Omega_s = \Omega_s' = 0$. The initial configuration is obtained by solving the Darwin-Riemann equilibrium equations (see LRS4) with $r/(a_1 + a_1') = 2.461$. In the second example, we have $n = n' = 0.5$, but now $M/M' = 1.2$ so that $R_o/M = 5$ and $R_o'/M' = 6$. Also we set $\Omega_s/(M/R_o^2)^{1/2} = 0.4$ (near the maximum possible value) and $\Omega_s' = 0$, corresponding to $\mathcal{C}/(M^3 R_o)^{1/2} = -0.1204$ and $\mathcal{C}' = 0$. The initial state has $r/(a_1 + a_1') = 2.385$ (and $r/R_o = 5$), and the nondimensional vorticity parameters are $f_R \equiv -(a_1^2 + a_2^2)\Lambda/(a_1 a_2 \Omega) = 3.818$, $f_R' = -2$. Also shown in Figure 3 are the energy curves of the constant$-\mathcal{C}$ equilibrium sequences. Initially, the binary closely follows the equilibrium sequence. As the dynamical instability develops, the radial velocity increases considerably, and thereafter the two stars merge hydrodynamically in just a few orbits. The terminal radial velocity at contact typically reaches 10% of the free-fall velocity. This qualitative behavior has already been observed in the simplified calculations we presented in LRS2 and LRS3, where the stars were constrained to retain their equilibrium shapes throughout the evolution.

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From Fig. 3, we also see that the tidal lag angle α or α' increases with decreasing r, attaining a large value of about $10°$ at binary contact. Let us consider the lag angle α in star M. In the absence of fluid viscosity, there are two contributions to this lag angle, $\alpha = \alpha_{dyn} + \alpha_{gr}$: one is the dynamical lag α_{dyn} discussed in §4, which arises from the rapid orbital decay (especially when the dynamical instability develops); the other is the *gravitational radiation dissipation lag* α_{gr} analogous to the usual viscous lag (see later). During the secular evolution phase (before the dynamical instability develops), the orbital decay timescale is

$$t_d \simeq \left| \frac{r}{\dot{r}} \right| \simeq \frac{5r^4}{64MM'M_t}, \tag{5.1}$$

where $M_t \equiv M + M'$. From equation (4.4), the dynamical lag is given by

$$\alpha_{dyn} \sim \frac{R_o^3 \Delta\Omega}{M} \left(\frac{64\,MM'M_t}{5\,r^4} \right) \simeq \frac{64}{5} \frac{R_o^3 M' M_t^{3/2}}{r^{11/2}}, \tag{5.2}$$

where in the second equality we have specialized in the case when $\Omega_s \simeq 0$ and $\Delta\Omega \simeq \Omega_{orb} \simeq (M_t/r^3)^{1/2}$. The gravitational radiation dissipation lag arises because gravitational radiation reaction directly exerts a torque on the star. From equation (3.4), this torque $\mathcal{N}_{gr} = \mathcal{F}_\phi = \partial\mathcal{W}/\partial\dot{\phi}$ is given by

$$\begin{aligned}
\mathcal{N}_{gr} &= -\frac{2}{5} \left(\frac{1}{5}\kappa_n M \right) \left[\math{I}_{\bar{1}\bar{2}}^{(5)} \cos 2\alpha + (\math{I}_{\bar{1}\bar{1}}^{(5)} - \math{I}_{\bar{2}\bar{2}}^{(5)})\frac{1}{2} \sin 2\alpha \right] (a_1^2 - a_2^2) \\
&\simeq -\frac{2}{5} \left(\frac{1}{5}\kappa_n M \right) \left(16\Omega_{orb}^5 \mu r^2 \cos 2\alpha \right) (a_1^2 - a_2^2),
\end{aligned} \tag{5.3}$$

where in the second equality we have used equation (3.12). Since the fluid circulation $\mathcal{C}$ in the star is conserved in the absence of viscosity, and since $|J_s| \simeq |\mathcal{C}|$ to the leading order (cf. eqs. [2.12], [2.17]), there must be a small misalignment α_{gr} of the tidal bulge so that $\mathcal{N}_{gr}$ can be balanced by the tidal torque $\mathcal{N}$ (cf. eq. [2.11]). Requiring $dJ_s/dt = \mathcal{N} + \mathcal{N}_{gr} = 0$, we obtain

$$\alpha_{gr} \simeq \frac{32}{15} \frac{MM_t^{3/2}}{r^{5/2}}, \tag{5.4}$$

which is independent of the radius and spin of the star. This result was derived previously in LRS5 (§9.2) for Roche-Riemann binaries. Although α_{gr} is larger than α_{dyn} for large binary separation $r \gtrsim 2R_o(M'/M)^{1/3}$, the radiation dissipation lag is always small ($\alpha_{gr} \lesssim 0.01$). At smaller orbital separation, the dynamical lag dominates. Moreover, as the dynamical instability develops, the orbital decay time becomes comparable to the orbital period, $t_d \sim t_{orb} = 1/\Omega_{orb}$, and the thus the lag angle near binary contact is

$$\alpha \simeq \alpha_{dyn} \sim \frac{\Omega_{orb}\Delta\Omega}{\omega_o^2} \sim \left(\frac{\Omega_{orb}}{\omega_o} \right)^2 \sim \left(\frac{M_t}{M} \right)^2 \left(\frac{R_o}{r} \right)^3. \tag{5.5}$$

For $M \sim M'$ and $r \simeq 2.6R_o$ (contact), we have $\alpha \sim 0.2$, in agreement with the numerical results of Fig. 3. Equation (5.5) implies that α is smaller for the spinning, more massive star, also in agreement with Fig. 3.

The gravitational wave forms can also be calculated using the standard quadrupole formula, giving

$$\begin{aligned}
h_+ &= -\frac{2}{D} \left[\mu r^2 \Omega_{orb}^2 \cos 2\theta + (I_{11} - I_{22})\Omega^2 \cos 2\phi + (I'_{11} - I'_{22})\Omega'^2 \cos 2\phi' \right] (1 + \cos^2\Theta), \\
h_\times &= -\frac{4}{D} \left[\mu r^2 \Omega_{orb}^2 \sin 2\theta + (I_{11} - I_{22})\Omega^2 \sin 2\phi + (I'_{11} - I'_{22})\Omega'^2 \sin 2\phi' \right] \cos\Theta,
\end{aligned} \tag{5.6}$$

where D is the source distance, and Θ specifies the angle between the direction of wave propagation and the z-axis. We have neglected higher-order terms which are smaller by a factor of order $|\dot{r}/r\Omega_{orb}|$.

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Equation (5.6) generalizes the expressions derived in LRS3 to the case when $\phi \neq \theta$ and $\phi' \neq \theta$ (cf. Fig. 1). Neglecting the small tidal correction, the wave frequency is $f \simeq (M + M')^{1/2} r^{-3/2}/\pi = 1123\ M_{1.4}^{-1}(R_o/5M)^{-3/2}(r/3R_o)^{-3/2}$ Hz (for $M = M'$). Figure 4 depicts the wave amplitude seen by an observer along the rotation axis ($\Theta = 0$) for one of the cases ($\Omega_s = \Omega'_s = 0$) shown in Figure 3. Comparing to the result for two-point masses, we see that a large phase error ($2-3$ cycles) develops during the last ~ 10 wave cycles as the result of the accelerated coalescence induced by the dynamical instability. Such signature should be detectable by specially configured ("dual recycled") interferometers that operate over adjustable, narrow bands around 1000 Hz (e.g., Strain & Meers 1991).

The dynamical effects of viscous dissipation can also be incorporated in the calculations. Dimensionally, the maximum possible value of viscosity is $\bar{\nu}_{max} \sim (MR_o)^{1/2}$. However, the viscosity due to electron-electron scattering (Flowers & Itoh 1979), which is the dominant source of microscopic viscosity since neutrons and protons are likely to be in superfluid states in cold coalescing neutron stars, is many orders of magnitude smaller than $\bar{\nu}_{max}$. Nevertheless, to illustrate the qualitative effects, we consider an extreme example, adopting $\bar{\nu} = 0.5R_o(M/R_o)^{1/2}$. In Figure 5, we compare neutron star binary evolution with and without viscosity. We see that viscous dissipation tends to synchronize the binary, i.e., to reduce Λ as compared to the inviscid cases. Such synchronization is the result of viscous tidal lag $\alpha_{vis} \sim \Delta\Omega/(\omega_o^2 t_{vis})$ (compare with eq. [4.4]), where $t_{vis} \sim R_o^2/\bar{\nu}$ is the viscous timescale (e.g., Zahn 1977; also see eq. [8.4] in LRS5). Also viscous dissipation tends to accelerate the coalescence. This is because orbital angular momentum is transferred to the stellar spin via viscous torque. However, as can be seen In Fig. 5(c), even with such an extreme value of viscosity, synchronization can never be achieved. In fact the binary always becomes more and more asynchronized as r decreases, even if it is corotating at large separation. This conclusion agrees with that of Bildsten & Cutler (1992).

Finally, it should be noted that our treatment so far has ignored post-Newtonian (PN) effects other than the lowest-order dissipative effect corresponding to the emission of gravitational radiation. However, for the typical value of $R_o/M = 5$, other PN effects are important and can alter the orbits considerably. In particular, even in the case of two point masses, these PN effects can by themselves make a circular orbit become unstable when the separation is smaller than some critical value ("inner-most stable orbit") r_{GR}. Kidder, Will & Wiseman (1992) have recently obtained $r_{GR} \simeq 6(M + M') + 4\mu$. The PN effects lead to a plunge orbit for $r < r_{GR}$, with significant infall radial velocity. For two point masses with $M = M'$, $r_{GR} \simeq 14M$. Since the Newtonian hydrodynamical stability limit is at $r_m \simeq 3R_o$ typically, r_{GR} and r_m are comparable for $R_o/M = 5$. Clearly, without including PN terms, our final coalescence trajectory can only be approximate. However, it is clear from our discussion that the Newtonian hydrodynamic effects discussed in this paper are at least as important as relativistic corrections to the orbital motion for the final phase of neutron star binary coalescence. When the two effects are both properly incorporated, the final coalescence is likely to be even faster, and assume a significant "head-on" character.

6. CONCLUSIONS

We have presented a simplified hydrodynamical treatment of close binary systems based on compressible ellipsoids obeying a polytropic equation of state. We employed this treatment to demonstrate the development of dynamical instability during the final phase of neutron star binary coalescence prior to contact. Such instability is accompanied by a significant radial infall velocity and dynamical tidal lag angles of order $10°$. We have also shown that the neutron star becomes more asynchronized as the binary orbit shrinks (see also Bildsten & Cutler 1992). Full hydrodynamical simulations of binary merger based on non-corotating models with large initial plunging velocity components (Shibata el al. 1992, 1993) show significantly different results from simulations based on corotating binary models without large infall velocities (Nakamura & Oohara 1991, Rasio & Shapiro 1992, 1994). However, these simulations should be considered only preliminary since the calculations start near contact, by which time dynamical effects are already important and the initial conditions in these simulations are very approximate.

Our dynamical binary models provide a new tool to study the pre-contact evolution of binary

neutron stars. The simplicity of replacing the full hydrodynamical equations with ODEs allows us to sample a large number of physical parameters (e.g., masses, spins, equations of states, etc.) and incorporate dissipative effects (gravitational radiation and viscosity) easily. Our Newtonian dynamical binary equations, when coupled with the post-Newtonian equations for point-mass binaries (e.g., Kidder et al. 1993), should give a reasonable description of the terminal phase of the binary prior to merger.

The numerical codes implementing the equations presented in this paper can be obtained from the authors upon request.

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Figure Captions

FIG. 1.— The coordinate system used for Darwin-Riemann binaries.

FIG. 2.— The development of the dynamical instability in a binary system with $p = M/M' = 1/2$, $n = n' = 0.5$, $K = K'$. The initial configurations are in equilibrium, and corotating. The left panels show the evolution of a dynamically unstable binary with $\hat{r} = r/(a_1 + a_1') = 1.17$, while the right panels show that of a stable binary with $\hat{r} = 1.18$. The solid lines correspond to star M, the dashed lines correspond to M'. Each line is labeled by the value of i in a_i.

FIG. 3.— The evolution of coalescing binaries driven by gravitational radiation. The system energy E, tidal angle α, radial velocity $v_r = \dot{r}$ and time t are shown as a function of binary separation r. Here $n = n' = 0.5$, $R_o/M = 5$, $R_o = R_o'$, and the calculations start ($t = 0$) at $r/R_o = 5$ and terminate when the surfaces of the stars contact. The left panels show the case with $M = M'$ and $\Omega_s = \Omega_s' = 0$; the right panels show the case with $M/M' = 1.2$, $\Omega_s/(M/R_o^3)^{1/2} = 0.4$ and $\Omega_s' = 0$. The dotted curve in the $E(r)$ diagram is the equilibrium energy of a constant-$\mathcal{C}$ binary sequence. The turning point marks the onset of dynamical instability. The dashed line in the right panel corresponds to α', while the solid line corresponds to α.

FIG. 4.— The waveform from neutron star binary coalescence shown in Fig. 3 ($\Omega_s = \Omega_s' = 0$). The dark solid line corresponds to zero viscosity, the light solid line assumes $\bar{\nu} = 0.5(MR_o)^{1/2}$. The dotted line is the result for two point masses.

FIG. 5.— Radial velocity v_r, number of orbital cycles N_{orb} and vorticity parameter Λ (measuring the degree of non-synchronization) of a coalescing neutron star binary. Here $M = M'$, $R_o/M = 5$ and $n = 0.5$. The solid lines are for $\mathcal{C} = \mathcal{C}' = 0$ (irrotation) with zero viscosity, the dashed lines for $\mathcal{C} = \mathcal{C}' = 0$ at $r/R_o = 5$ and with $\bar{\nu} = 0.5(MR_o)^{1/2}$; the long-dashed lines are for $\Lambda = \Lambda' = 0$ (corotation) at $r/R_o = 5$ with no viscosity, and the dotted-dashed lines are for $\bar{\nu} = 0.5(MR_o)^{1/2}$. The dotted lines in (a)-(b) are the results for two point masses.

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